A General Formulation for Librational Dynamics of Spacecraft with Deploying Appendages

V.J. Modi* and A.M. Ibrahim†
University of British Columbia, Vancouver, Canada

This paper presents a general formulation for studying librational dynamics of a large class of spacecraft during deployment of flexible members. It is applicable to a variety of missions ranging from deployment of antennas, booms, and solar panels from a spacecraft to manufacturing of trusses for space platforms using the Space Shuttle. The governing nonlinear, nonautonomous, and coupled equations of motion are extremely difficult to solve even with the help of a computer, not to mention the cost involved. The equations are being so programmed on an AMDAHL 470-V8 digital computer to help assess the effect of shifting center of mass, nonlinearities, change in inertia of the central rigid body during deployment, number of admissible functions, etc., which complicate the problem considerably. The formulation is ideally suited to help assess the effect of complex interactions between flexibility, deployment, attitude dynamics, and stability for a large family of spacecraft for the present, as well as the next, generation.

Nomenclature

A_i^p	= area of the ith plate type appendage
B_j	$= M_j^b/M$
$\bar{B}_j, \bar{B}_{r,j}$	= position vectors, Fig. 5
C	= instantaneous center of mass of satellite
C_0	= center of mass of satellite before deployment
$\bar{C}, \bar{C}_0$	= position vectors, Fig. 5
d_i^p	= width of the ith plate type appendage
$D_j^b, D_i^p, D_r,$ $D_{r,j}^b, D_{r,i}^p$	= domains associated with rigid and flexible components of satellite, Fig. 4
e	= orbit eccentricity
e_ζ, e_η, e_ξ	= unit vectors along ζ, η, ξ directions, respectively
E	= center of mass of the Earth
$\bar{f}, \bar{g}$	= position vectors to mass elements of flexible and rigid (i.e., undeployed) parts of an appendage, respectively
F_q	= generalized force associated with the generalized coordinate q
G	= flexural rigidity
h	= angular momentum per unit mass of satellite
$\{H\}$	= angular momentum due to deploying and vibrating appendages
H_{mn}	= generalized coordinate associated with ϕ_m, ψ_n modes of free-free and fixed-free beams, respectively, to represent plate type appendage oscillations, Eq. (7)
$\bar{i}, \bar{j}, \bar{k}$	= unit vectors along x, y, z axes, respectively
$[I]$	= satellite inertia diadic
L^p, L^b	= instantaneous length of a deploying plate and beam, respectively
$\{l\}$	$\equiv \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$ = direction cosines of the unit vector along $\bar{R}_c$ with respect to x, y, z axes, respectively
M	= total mass of satellite
M_r	= mass of the central rigid body of satellite
M^b, M^p	= mass of the flexible (deployed) part of beam and plate type appendages, respectively
M_r^b, M_r^p	= mass of the rigid (undeployed) part of beam and plate type appendages, respectively
N_b, N_p	= number of beam and plate type appendages, respectively
O_r	= center of mass of the central rigid body of satellite (excluding appendages)
$\bar{p}_i, \bar{p}_{r,i}$	= position vectors, Fig. 5
P_l, Q_l	= generalized coordinates associated with lth mode of a fixed-free beam vibration, Eq. (8)
$\bar{r}$	= position vector of a mass element dM in any domain with respect to x, y, z axes
$\bar{r}_r$	= position vector to a mass element dM_r of the central rigid body with respect to x_r, y_r, z_r axes
$\bar{R}$	= position vector to a mass element dM of satellite from the Earth's center
$\bar{R}_c$	= position vector from the Earth's center to the instantaneous center of mass C
$[S], [T]$	= transformation matrix defining rotation between ζ, η, ξ and x, y, z axes, Eq. (3)
t	= time
T	= kinetic energy
u	= vibrational displacement of a plate element
v, w	= transverse displacement of a beam element along ζ, η directions, respectively
U	= gravitational potential energy
V	= strain energy
x, y, z	= body coordinates with origin at C, Fig. 4
$x_r^b, y_r^b, z_r^b;$ $x_r^p, y_r^p, z_r^p;$ x_r, y_r, z_r	= body coordinates associated with rigid parts of beam, plate, and central satellite body, respectively, Fig. 4
X, Y, Z	= inertial coordinate system with origin at E
X_0, Y_0, Z_0	= orbital coordinate system with origin at C_0; Y_0 along local vertical, Z_0 along local horizontal, and X_0 aligned with orbit normal
α, β, γ	= pitch, yaw, and roll librational angles
ζ, η, ξ	= local coordinates associated with a deploying appendage, Fig. 4
θ	= true anomaly, $\dot{\theta} = \Omega$
μ_e	= universal gravitational constant
ϕ_m, ψ_n	= modes of free-free and fixed-free beams, respectively
$\{\omega\}$	= satellite angular velocity vector

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*Professor, Department of Mechanical Engineering. Associate Fellow AIAA.

†Graduate Research Fellow, Department of Mechanical Engineering.

Superscripts, Subscripts, and Miscellaneous Symbols

$(\bar{\ })$ = vector

$(\dot{\ })$, $d(\ )/dt$ = time rate of change in inertial and reference coordinate systems for vectorial quantity, respectively

$[\ ]$ = matrix

$\{\ \}$ = vector

$d(\ )$ = differential element

$(\ )_j^b$ = quantity associated with flexible (deployed) part of the jth beam

$(\ )_i^p$ = quantity associated with flexible (deployed) part of the ith plate

$(\ )_{r,j}^b, (\ )_{r,i}^p$ = quantities associated with rigid (undeployed) parts of the jth beam and ith plate, respectively

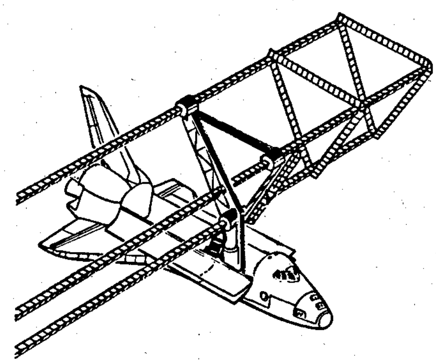

Fig. 1 Artist's view of the Space Shuttle-based manufacture of structural components for construction of space platforms.

Introduction

IN the early stages of space exploration when spacecraft tended to be small, mechanically simple, and essentially inflexible, elastic deformations were relatively insignificant. Numerous investigations involving active and passive stabilization procedures and accounting for internal as well as external forces have been carried out assuming satellites to be rigid.[1] However, in a modern space vehicle carrying lightweight deployable members, which are inherently flexible, it is no longer true. This aspect can be emphasized through several illustrations:

1) Ever-increasing demand on power for operation of the onboard instrumentation, scientific experiments, communications systems, etc., has been reflected in the size of the solar panels. The Canada/USA Communications Technology Satellite (CTS, Hermes) launched in January 1976 carried two solar panels, 1.14 × 7.32 m each, to generate around 1.2 kW.

2) Use of the large members may be essential in some missions. For example, the Radio Astronomy Explorer (RAE) satellite used four 228.2-m antennas to detect low-frequency signals.

3) For identifying extraterrestrial radio sources, the Applied Physics Laboratory of the Johns Hopkins University once proposed a gravitationally stabilized Tethered Orbiting Interferometer (TOI), consisting of two spacecraft connected by a line 2 to 6 km long. In fact, NASA has shown considerable interest in exploiting applications of the Space Shuttle-based tethered subsatellite system extending to 100 km, and has initiated preliminary studies through contracts to establish its feasibility.

4) Preliminary configurations of the next generation of satellites suggest a trend towards spacecraft with large flexible members.

5) Space engineers are involved in assessing feasibility of construction of gigantic space stations which cannot be launched in their entirety from the Earth, but must be constructed in space through integration of modular subassemblies. The concept of Satellite Solar Power Station (SSPS) in the geostationary orbit as proposed by Glaser[2] and in-orbit assembly of enormous orbiting stations[3-6] such as the Space Operations Center (SOC) suggest an increasing role of structural flexibility in their dynamical and control considerations (Fig. 1).

This being the case, flexibility effects on satellite attitude motion and its control have become a topic of considerable investigation. Over the years, a large body of literature pertaining to the various aspects of satellite system response, stability, and control has evolved which has been reviewed quite effectively in Refs. 7-13.

It should be emphasized that prediction of satellite attitude motion is by no means a simple proposition, even if the system is rigid. Flexible character of the appendages makes the problem enormously complex. It is, therefore, understandable why transient behavior associated with the critical phase of attitude acquisition and/or deployment-related maneuvers has received relatively little attention. On the other hand, although the deployment effects are of a transient nature, they may be felt over a long period of time as a result of relatively small extension rates that are normally associated with large appendages. The Space Shuttle-based tethered system mentioned before may take 6 to 8 hours to deploy and much longer to retrieve.[14]

The complex character of the problem has led to analyses which often involve simplifying assumptions. Lang and Honeycutt[15] as well as Cloutier[16] studied the problem of deployment dynamics representing an appendage by point masses. On the other hand, there are several efforts reported in the literature in which authors have treated flexible members as rigid bodies.[17-19] Cherchas and his associates[20,21] as well as Dow et al.[22] did consider flexible membrane or beam type appendages but with a specific configuration. Furthermore, the appendages were considered to be uniform with a fixed deployment velocity. More recently, Jankovic[23] investigated dynamics of the CTS solar panels during deployment and correlated measured tip acceleration with the analytical prediction. The librational dynamics of a body deploying two plate type flexible members normal to the orbital plane was studied by Ibrahim and Misra.[24] The effect of deployment velocities and plate properties on the librational response was investigated. A rather general formulation for the class of satellites with flexible deploying beam type appendages has been presented by Lips and Modi.[25-27] Interaction between the libration dynamics, flexibility, and deployment was studied and it was noted that instability may result under certain combinations of system parameters.

Flexible character of the appendages renders the system hybrid, i.e., the system is described in terms of discrete and distributed coordinates. The resulting governing equations of motion in general do not admit to any closed-form solution. They are normally transformed into a set of ordinary differential equations using finite element, lumped parameter, or assumed mode methods with generalized coordinates depending on time alone. Here an assumed mode discretization procedure is favored, as the elastic appendage displacements can be represented adequately using relatively few equations. Hence displacement of a flexible member is described by a linear combination of space-dependent admissible functions multiplied by time-dependent generalized coordinates.

With this as background, the paper attempts to develop a rather general formulation for a librating system deploying flexible appendages with plate or beam type behavior. The formulation for such a system with time-dependent inertia is complicated by the shifting center of mass, changing central rigid body inertia, offset of the appendage attachment point, number of admissible functions, etc. It is proposed here to formulate the problem in a manner that would enable a systematic assessment of the influence of the parameters mentioned with a possibility of simplifying the equations at least for the configurations likely to be encountered in this decade. Past experience with relatively simple nonlinear, nonautonomous, and coupled systems suggests that their solution, even with the help of a computer, is quite demanding and exorbitant in cost. Any realistic simplification of the problem is indeed desirable.

The basic system is so complex that to check the validity of the governing equations and associated program for their integration presents a challenging task. The formulation thus can serve as a comparative validating scheme in association with other formulations when they become available. The dynamic equations would also serve as a basis for assessing the effect of environmental forces and development of control strategies.

Essential features of the general formulation may be summarized as follows:

1) Satellite of arbitrary geometry in a general orbit undergoing three-axis librations;

2) Arbitrary number and orientation of beam and/or plate type flexible appendages deploying independently at arbitrary velocity and acceleration;

3) Appendage with variable mass density, flexural rigidity, and cross-sectional area along its length;

4) Governing nonlinear equations, accounting for gravitational effects, shifting center of mass, changing rigid body inertia, and appendage offset, together with transverse oscillations;

5) Modified Eulerian rotations, so chosen as to make the governing equations applicable to both spin-stabilized and gravity gradient orientations;

6) Equations programmed in nonlinear as well as linearized forms to permit the study of both large angle maneuvers and nonlinear effects.

Spacecraft Model

Consider a satellite of mass M (Fig. 2) consisting of a central rigid body deploying an arbitrary number of plate and/or beam type flexible appendages in any given direction. The satellite is free to negotiate a specified trajectory. Let the position vector $\bar{R}_c$ and true anomaly θ define the location of the instantaneous center of mass C of the spacecraft with respect to the inertial coordinate system X, Y, Z having its origin at the center of the Earth E. C_0 represents location of C without any vibrations or asymmetric deployment. An orthogonal orbiting reference frame X_0, Y_0, Z_0 with its origin at C_0 is so oriented that Y_0 and Z_0 are along the local vertical and horizontal, respectively, while X_0 is aligned with the orbit normal. The body coordinates x, y, z with origin at C coincide with the orbital coordinates X_0, Y_0, Z_0 in the absence of any librations and vibrations.

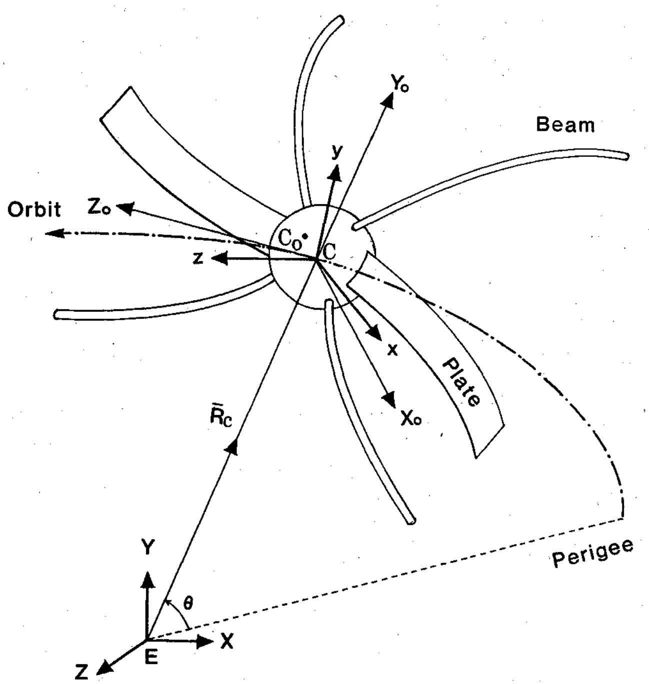

Fig. 2 Geometry of orbiting spacecraft with flexible deploying beam and plate type appendages.

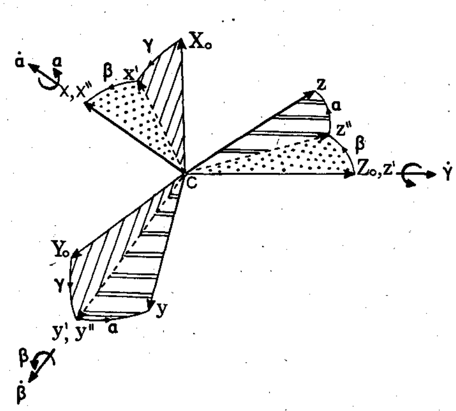

Fig. 3 Eulerian rotations showing roll (γ), yaw (β), and pitch (α) librations.

The orientations of the body axes x, y, z at any instant t relative to the orbital coordinate frame X_0, Y_0, Z_0 can be described by a set of modified Eulerian rotations as follows: γ (roll) about Z_0 giving x', y', z'; β (yaw) about y' resulting in x'', y'', z''; and finally α (pitch) about x'' yielding x, y, z (Fig. 3). The angular velocity in the body coordinate system can readily be written in terms of librational angles and velocities as:

$$\omega_x = \dot{\alpha} - \dot{\gamma}\sin\beta + \Omega\cos\beta\cos\gamma \tag{1a}$$

$$\omega_y = \dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\cos\beta + \Omega(-\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) \tag{1b}$$

$$\omega_z = -\dot{\beta}\sin\alpha + \dot{\gamma}\cos\alpha\cos\beta + \Omega(\sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma) \tag{1c}$$

where Ω is the orbital angular velocity.

The spacecraft components of interest are schematically sketched in Fig. 4. The rigid part of the spacecraft is represented by domain D_r and its mass by M_r. At an instant during the deployment phase, the undeployed rigid parts of the ith plate and the jth beam are represented by $D^p_{r,i}$ and $D^b_{r,j}$, and their masses by $M^p_{r,i}$ and $M^b_{r,j}$, respectively, with $i = 1,2,\ldots,N_p$; $j = 1,2,\ldots,N_b$. Here N_p and N_b are the number of plate and beam type appendages, respectively, carried by the satellite (for clarity, only one plate and one beam are shown in Fig. 4). The flexible deployed domains with masses M^p_i and M^b_j are denoted by D^p_i and D^b_j for the ith plate and jth beam, respectively. The total mass M of the satellite is thus given by

$$M = M_r + \sum_{i=1}^{N_p} (M^p_{r,i} + M^p_i) + \sum_{j=1}^{N_b} \left(M^b_{r,j} + M^b_j\right) \tag{2a}$$

or in a nondimensional form

$$1 = R_r + \sum_{i=1}^{N_p} (P_{r,i} + P_i) + \sum_{j=1}^{N_b} (B_{r,j} + B_j) \tag{2b}$$

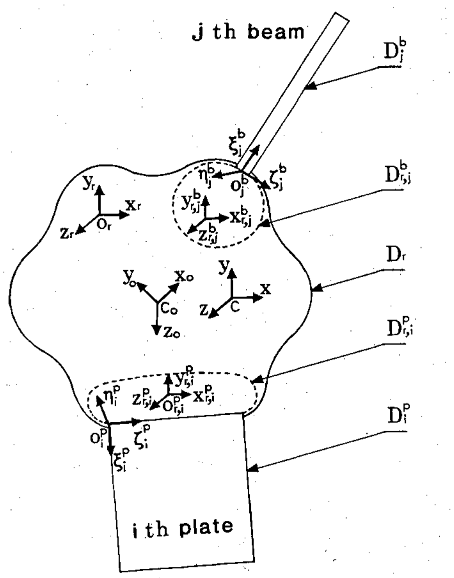

Fig. 4 Domains associated with rigid and flexible components of spacecraft and corresponding reference coordinate systems.

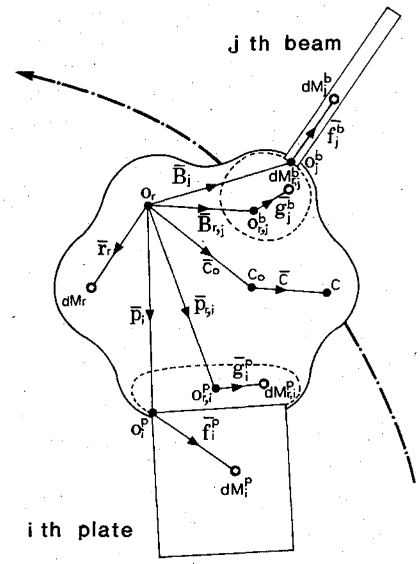

Fig. 5 Schematic diagram showing position vectors to mass elements in different domains.

where

$$R_r = M_r/M; \qquad P_{r,i} = M^p_{r,i}/M; \qquad P_i = M^p_i/M$$

$$B_{r,j} = M^b_{r,j}/M \quad \text{and} \quad B_j = M^b_j/M$$

Fixed to the center of masses of the rigid domains D_r, $D^p_{r,i}$, and $D^b_{r,j}$ are the local coordinate axes x_r, y_r, z_r; x^p_i, y^p_i, z^p_i, and x^b_j, y^b_j, z^b_j, which are parallel to the body axes x, y, z. At any instant, orientation of a given plate (domain D^p_i) with respect to its nominal undeflected configuration is denoted by a body-fixed coordinate system $\zeta^p_i, \eta^p_i, \xi^p_i$, (Fig. 4). Here η^p_i is normal to the nominal plate plane, ξ^p_i along the direction of deployment, and ζ^p_i normal to ξ^p_i in the plane of the plate. Similarly $\zeta^b_j, \eta^b_j, \xi^b_j$ define orientation of the beams.

The directions of the coordinate axes $\zeta^p_i, \eta^p_i, \xi^p_i$, and $\zeta^b_j, \eta^b_j, \xi^b_j$ are related to the body axes x, y, z through the transformation

$$\begin{Bmatrix} \zeta^p_i \\ \eta^p_i \\ \xi^p_i \end{Bmatrix} = [T_i] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \zeta^b_j \\ \eta^b_j \\ \xi^b_j \end{Bmatrix} = [S_j] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{3}$$

respectively, where $[T_i]$ and $[S_j]$ are direction cosine matrices.

Let $\bar r_r$, $\bar g^p_i$, $\bar g^b_j$, $\bar f^p_i$, and $\bar f^b_j$ define the position of the differential mass elements $\mathrm{d}M_r$, $\mathrm{d}M^p_{r,i}$, $\mathrm{d}M^b_{r,j}$, $\mathrm{d}M^p_i$, and $\mathrm{d}M^b_j$ in domains D_r, $D^p_{r,i}$, $D^b_{r,j}$, D^p_i, and D^b_j, respectively (Fig. 5). Clearly,

$$\bar r_r = x_r \bar i + y_r \bar j + z_r \bar k \tag{4a}$$

$$\bar g^p_i = x^p_i \bar i + y^p_i \bar j + z^p_i \bar k \tag{4b}$$

$$\bar g^b_j = x^b_j \bar i + y^b_j \bar j + z^b_j \bar k \tag{4c}$$

$$\bar f^p_i = \zeta^p_i \bar e^p_{\zeta i} + u_i(\zeta^p_i, \xi^p_i, t) \bar e^p_{\eta i} + \xi^p_i \bar e^p_{\xi i} \tag{4d}$$

$$\bar f^b_j = v_j(\xi^b_j, t) \bar e^b_{\zeta j} + w_j(\xi^b_j, t) \bar e^b_{\eta j} + \xi^b_j \bar e^b_{\xi j} \tag{4e}$$

where u_i is the vibration displacement of the element $\mathrm{d}M^p_i$ and $\bar e^p_{\zeta i}$, $\bar e^p_{\eta i}$, and $\bar e^p_{\xi i}$ are unit vectors along $\zeta^p_i, \eta^p_i, \xi^p_i$ axes, respectively. v_j and w_j are the components of vibrational displacement of the element of mass $\mathrm{d}M^b_j$ on the jth beam, and $\bar e^b_{\zeta j}$, $\bar e^b_{\eta j}$, and $\bar e^b_{\xi j}$ are unit vectors along $\zeta^b_j, \eta^b_j, \xi^b_j$ axes, respectively. Note that both the beam and plate appendages are restricted to undergo transverse oscillations only, i.e., the axial component of the deformation is taken to be negligible.

Because of the deployment of appendages and their vibrations, the center of mass of the spacecraft, as well as its inertia diadic, will be functions of time. The vector $\bar C$ keeps track of the position of the instantaneous center of mass C relative to point C_0. By definition, position of the instantaneous center of mass is given by,

$$\begin{aligned} M(\bar C_0 + \bar C) = {} & \int_{M_r} \bar r_r \,\mathrm{d}M_r \\ & + \sum_{i=1}^{N_p} \left[\int_{M^p_{r,i}} (\bar p_{r,i} + \bar g^p_i)\,\mathrm{d}M^p_{r,i} + \int_{M^p_i} (\bar p_i + \bar f^p_i)\,\mathrm{d}M^p_i \right] \\ & + \sum_{j=1}^{N_b} \left[\int_{M^b_{r,j}} (\bar B_{r,j} + \bar g^b_j)\,\mathrm{d}M^b_{r,j} + \int_{M^b_j} (\bar B_j + \bar f^b_j)\,\mathrm{d}M^b_j \right] \end{aligned} \tag{5}$$

Here, $\bar p_{r,i}$, $\bar p_i$, $\bar B_{r,j}$, and $\bar B_j$ define the locations of points $O^p_{r,i}$, O^p_i, $O^b_{r,j}$, and O^b_j, with respect to point O_r, respectively (Fig. 5).

The inertia diadic can be written as,

$$\begin{aligned} [I] &= \int_M \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \mathrm{d}M \\ &= \int_M [\bar r \cdot \bar r [E] - \bar r \bar r]\,\mathrm{d}M \end{aligned} \tag{6}$$

where $[E]$ is a unit matrix and $\bar r$ is the position vector of an element $\mathrm{d}M$ relative to the instantaneous center of mass C.

It is convenient to represent the vibrational displacements in terms of a set of admissible functions which can be somewhat arbitrary as long as they satisfy at least the geometric boundary conditions. The exact modes of vibration of a rectangular plate clamped at one side and free at the other three sides are not known. Hence, u for a given plate was assumed to be of the form

$$u(\zeta^p,\xi^p,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}H_{mn}(t)\phi_m(\zeta^p)\psi_n(\xi^p) \tag{7}$$

However, a beam type appendage is free to undergo transverse oscillations v, w in two orthogonal directions ζ and η, respectively. They are represented by

$$v(\xi^b,t)=\sum_{l=1}^{\infty}P_l(t)\psi_l(\xi^b) \tag{8a}$$

and

$$w(\xi^b,t)=\sum_{l=1}^{\infty}Q_l(t)\psi_l(\xi^b) \tag{8b}$$

Here ψ_n and ϕ_m were chosen to be the characteristic functions of a fixed-free and free-free beam, respectively, and are given by

$$\begin{aligned}\psi_n(\xi)&=[\cosh(\lambda_n\xi/L)-\cos(\lambda_n\xi/L)]\\&\quad-\sigma_m[\sinh(\lambda_n\xi/L)-\sin(\lambda_n\xi/L)]\\&\qquad n=1,2,\ldots,\infty;\ (L=L^p \text{ or } L^b)\\ \phi_1(\zeta^p)&=1 \text{ (rigid body translation)}\\ \phi_2(\zeta^p)&=\sqrt{3}\,(1-2\zeta^p/d) \text{ (rigid body rotation)}\\ \phi_m(\zeta^p)&=[\cosh(\mu_n\zeta^p/d)+\cos(\mu_n\zeta^p/d)]\\&\quad-\delta_n[\sinh(\mu_n\zeta^p/d)+\sin(\mu_n\zeta^p/d)]\\&\qquad m=n+2;\ n=1,2,\ldots,\infty\end{aligned}$$

Here d and L^p are the width and length, respectively, of the plate type appendage under consideration and λ_m, μ_n are the roots of the transcendental equations

$$1+\cosh\lambda\cos\lambda=0$$

and

$$1-\cosh\mu\cos\mu=0$$

Note that σ_m and δ_n are constants dependent on λ_m and μ_n, respectively, and H_{mni}, P_{lj}, Q_{lj} are the unknown generalized coordinates associated with the vibration degrees of freedom.

Development of the Equations of Motion

In order to derive the equations of motion, expressions for the kinetic and gravitational potential energy of the spacecraft as well as the strain energy stored in the appendages were first obtained. The kinetic energy of the system can be written as

$$T=\frac{1}{2}\int_M\dot{\bar{R}}\cdot\dot{\bar{R}}\,\mathrm{d}M \tag{9}$$

where $\bar{R}$ defines the position of a differential element of mass $\mathrm{d}M$ with respect to the center of the Earth E. If $\bar{r}$ is the position vector of the element with respect to the body coordinate system x, y, z, then

$$\bar{R}=\bar{R}_c+\bar{r} \tag{10}$$

where $\bar{r}$ for different domains is defined as

$$\begin{aligned}\bar{r}&=-\bar{C}-\bar{C}_0+\bar{r}_r && D_r \text{ domain}\\&=-\bar{C}-\bar{C}_0+\bar{p}_r+\bar{g}^p && D_r^p \text{ domain}\\&=-\bar{C}-\bar{C}_0+\bar{B}_r+\bar{g}^b && D_r^b \text{ domain}\\&=-\bar{C}-\bar{C}_0+\bar{p}+\bar{f}^p && D^p \text{ domain}\\&=-\bar{C}-\bar{C}_0+\bar{B}+\bar{f}^b; && D^b \text{ domain}\end{aligned}$$

Differentiating Eq. (10) with respect to time

$$\dot{\bar{R}}=\dot{\bar{R}}_c+\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}+(\bar{\omega}\times\bar{r}) \tag{11}$$

Substituting Eq. (11) into Eq. (9) and noting that C is the center of mass of the system

$$\begin{aligned}T&=\frac{1}{2}M\dot{\bar{R}}_c\cdot\dot{\bar{R}}_c+\frac{1}{2}\int_M(\bar{\omega}\times\bar{r})\cdot(\bar{\omega}\times\bar{r})\,\mathrm{d}M\\&\quad+\int_M\left[\frac{1}{2}\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}+\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\cdot(\bar{\omega}\times\bar{r})\right]\mathrm{d}M\end{aligned} \tag{12}$$

It may be noted that for the rigid parts $\dot{\bar{r}}=-\dot{\bar{C}}$. For a mass element in the flexible plate $\dot{\bar{r}}=-\dot{\bar{C}}+\dot{\bar{f}}^p$, and for the beam $\dot{\bar{r}}=-\dot{\bar{C}}+\dot{\bar{f}}^b$ are obtained through differentiation of Eqs. (4d) and (4e), respectively

$$\frac{\mathrm{d}\bar{f}^p}{\mathrm{d}t}=\left(\frac{\partial u}{\partial t}+\dot{L}^p\frac{\partial u}{\partial\xi^p}\right)\bar{e}_\eta^p+\dot{L}^p\bar{e}_\xi^p$$

$$\frac{\mathrm{d}\bar{f}^b}{\mathrm{d}t}=\left(\frac{\partial v}{\partial t}+\dot{L}^b\frac{\partial v}{\partial\xi^b}\right)\bar{e}_\zeta^b+\left(\frac{\partial w}{\partial t}+\dot{L}^b\frac{\partial w}{\partial\xi^b}\right)\bar{e}_\eta^b+\dot{L}^b\bar{e}_\xi^b$$

where $\dot{L}^p$ is the deployment rate of the plate and $\dot{L}^b$ of the beam type appendage. Note the convective components associated with deployment rates. Deformation velocity components for a mass element on a plate or a beam can be obtained from Eqs. (7) and (8) as

$$\begin{aligned}\frac{\partial u}{\partial t}&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\dot{H}_{mn}\phi_m(\zeta^p)\psi_n(\xi^p,L^p)\\&\quad-H_{mn}\phi_m(\zeta^p)\frac{\mathrm{d}\psi_n}{\mathrm{d}\xi^p}\frac{\dot{L}^p\xi^p}{L^p}\Big]\end{aligned}$$

Note that now ψ_n is a function of both ξ and L.

$$\frac{\partial v}{\partial t}=\sum_{l=1}^{\infty}\left[\dot{P}_l\psi_l(\xi^b,L^b)-P_l\frac{\mathrm{d}\psi_l}{\mathrm{d}\xi^b}(\dot{L}^b\xi^b/L^b)\right]$$

$$\frac{\partial w}{\partial t}=\sum_{l=1}^{\infty}\left[\dot{Q}_l\psi_l(\xi^b,L^b)-Q_l\frac{\mathrm{d}\psi_l}{\mathrm{d}\xi^b}(\dot{L}^b\xi^b/L^b)\right]$$

Note, the second term in each of the above equations arises because ψ_n depends on the time-dependent length of the deploying appendages. Substitution of the above relations in Eq. (12), followed by some algebraic manipulations, leads to

$$T=T_{\mathrm{orb}}+T_{\mathrm{vib}}+\{\omega\}^T\{H\}+\tfrac{1}{2}\{\omega\}^T[I]\{\omega\} \tag{13}$$

where T_{orb} = Kinetic Energy (KE) due to orbital motion; T_{vib} = KE component due to pure vibration; $\frac{1}{2}\{\omega\}^T[I]\{\omega\}$ = KE due to pure rotation; $\{\omega\}^T\{H\}$ = KE due to coupling between vibrational and rotational modes; $\{\omega\}$ = angular velocity vector; $\{H\}$ = angular momentum with respect to body frame due to deploying and vibrating appendages; and $[I]$ = time-dependent inertia matrix.

The gravitational potential energy of the satellite can be written as

$$U = -(\mu_e/R_c) - (\mu_e/2R_c^3)\big[I_{xx}(1-3l^2) + I_{yy}(1-3m^2) + I_{zz}(1-3n^2) + 6(lmI_{xy} + mnI_{yz} + nlI_{zx})\big]$$
$$= -(\mu_e/R_c) - (\mu_e/2R_c^3)\mathrm{tr}[I] + (3\mu_e/2R_c^3)\{l\}^T[I_n]\{l\}$$

Here the first term represents potential energy due to the satellite treated as a point mass and the rest of the expression is the contribution due to its rotation.

The strain energy stored in the appendages during their vibration is given by

$$V = \sum_{i=1}^{N_p} V_i^p + \sum_{j=1}^{N_b} V_j^b$$

where

$$V_i^p = \frac{1}{2}G_{\eta,i}^p\int_{A_i^p}\left[\left[\frac{\partial^2 u_i}{\partial\zeta_i^{p^2}}\right]^2 + 2\nu_i\frac{\partial^2 u_i}{\partial\zeta_i^{p^2}}\frac{\partial^2 u_i}{\partial\xi_i^{p^2}} + \left[\frac{\partial^2 u_i}{\partial\xi_j^{p^2}}\right]^2 + 2(1-\nu_i)\left[\frac{\partial^2 u_i}{\partial\zeta_i^p\partial\xi_i^p}\right]^2\right]\mathrm{d}A_i^p$$

$$V_j^b = \frac{1}{2}\int_{L_j^b}\left[G_{\zeta,j}^b\left[\frac{\partial^2 v_j}{\partial\xi_j^{b^2}}\right]^2 + G_{\eta,j}^b\left[\frac{\partial^2 w_j}{\partial\xi_j^{b^2}}\right]^2\right]\mathrm{d}L_j^b$$

here $G_{\eta,i}^p$ and ν_i are the flexural rigidity and Poisson's ratio, respectively, of the ith plate while $G_{\zeta,j}^b$ and $G_{\eta,j}^b$ represent the flexural rigidity of the ith beam in ζ_j^b and η_j^b directions, respectively. Thus, the total strain energy expression becomes

$$V = \frac{1}{2}\sum_{i=1}^{N_p} G_{\eta,i}^p\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{mni}^2\left(\frac{\lambda_m^4}{(L_i^p)^4} + \frac{\mu_n^4}{(d_i^p)^4}\right)L_i^p d_i^p + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{H_{mni}H_{rsi}}{L_i^p d_i^p} \times\left(2\nu_i J_{n,s}^1 J_{m,r}^3 + 2(1-\nu_i)J_{n,s}^2 J_{m,r}^4\right)\right] + \frac{1}{2}\sum_{j=1}^{N_b}\sum_{l=1}^{\infty}\left[G_{\zeta,j}^b P_{lj}^2 + G_{\eta,j}^b Q_{lj}^2\right]\left(\frac{\lambda_l^4}{(L_j^b)^3}\right)$$

where $J_{n,s}^1$, $J_{n,s}^2$, $J_{m,r}^3$, and $J_{m,r}^4$ are constants given by

$$J_{n,s}^1 = L_i^p\int^{L_i^p}\frac{\mathrm{d}^2\psi_n}{\mathrm{d}\xi_{pi}^2}\psi_s\,\mathrm{d}\xi_i^p$$

$$J_{n,s}^2 = L_i^p\int^{L_i^p}\frac{\mathrm{d}\psi_n}{\mathrm{d}\xi_{pi}}\frac{\mathrm{d}\psi_s}{\mathrm{d}\xi_{pi}}\,\mathrm{d}\xi_i^p$$

$$J_{m,r}^3 = d_i^p\int_0^{d_i^p}\frac{\mathrm{d}^2\phi_m}{\mathrm{d}\zeta_{pi}^2}\phi_r\,\mathrm{d}\zeta_i^p$$

$$J_{m,r}^4 = d_i^p\int_0^{d_i}\frac{\mathrm{d}\phi_m}{\mathrm{d}\zeta_{pi}}\frac{\mathrm{d}\phi_r}{\mathrm{d}\zeta_{pi}}\,\mathrm{d}\zeta_i^p$$

This can be written in a compact matrix form as

$$V = \{q\}^T[V]\{q\}$$

where $\{q\}$ represents generalized coordinates associated with the vibration degrees of freedom,

$$\{q\} = \begin{Bmatrix} H_{mni} \\ \cdots \\ P_{lj} \\ \cdots \\ Q_{lj} \end{Bmatrix}$$

and $[V]$ is a symmetric matrix of vibration modes used to represent flexural deformations of the flexible appendages.

Using the Lagrangian procedure the governing equations of motion can now be obtained from,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial(U+V)}{\partial q} = F_q$$

where

$$q = R_c, \theta, \alpha, \beta, \gamma, H_{mni}, P_{lj}, Q_{lj}$$
$$m = 1,2,\ldots,p \qquad n = 1,2,\ldots,r$$
$$i = 1,2,\ldots,N_p \qquad l = 1,2,\ldots,s$$
$$j = 1,2,\ldots,N_b$$

hence the satellite dynamics is simulated by a system of q second order ordinary differential equations where

$$q = 5 + (p\times r\times N_p) + 2(s\times N_b)$$

The librational motion of the system and the vibrations of the appendages normally have very little effect on the orbital motion unless the system dimensions become comparable to the position vector R_c. Hence, for most studies, the orbit can be computed using the classical Keplerian relations

$$R_c = h^2/\mu_e(1 + e\cos\theta), \qquad R_c^2\dot{\theta} = h$$

where h is the angular momentum per unit mass of the satellite and e the eccentricity of the orbit. The equations of motion in the libration degrees of freedom are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\{\frac{\partial\omega}{\partial\dot{q}}\right\}^T\{H\}\right) - \left\{\frac{\partial\omega}{\partial q}\right\}^T\{H\} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\left\{\frac{\partial\omega}{\partial\dot{q}}\right\}^T[I]\{\omega\}\right) - \left\{\frac{\partial\omega}{\partial q}\right\}^T[I]\{\omega\} + \frac{\partial U}{\partial q} = Q_q, \qquad q = \alpha,\beta,\gamma$$

The equations of motion in the vibrational degrees of freedom can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_{\mathrm{vib}}}{\partial\dot{q}}\right) - \frac{\partial T_{\mathrm{vib}}}{\partial q} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\left\{\frac{\partial H}{\partial\dot{q}}\right\}^T\{\omega\}\right) - \left\{\frac{\partial H}{\partial q}\right\}^T\{\omega\} - \frac{1}{2}\{\omega\}^T\frac{\partial}{\partial q}[I]\{\omega\} + \frac{\partial(U+V)}{\partial q} = Q_q$$

$$q = H_{mni}, P_{lj}, Q_{lj}$$

The number of equations representing vibrational motion would depend on the number of modes used

$$N = \text{number of vibrational equations}$$
$$= (p \times r \times N_p) + 2(s \times N_b)$$

Computational Considerations

The governing nonlinear, nonautonomous, and coupled equations of motion are extremely difficult to solve even with the help of a computer, not to mention the cost involved. They are being programmed for numerical integration using an AMDAHL 470-V8 digital computer. The integration routine (GEARB-I.M.S.L.) is based on the implicit Adams' method with built-in error control. The main objective is to assess the effect of flexibility and deployment on the attitude dynamics and stability of satellites and space stations.

The dynamics of the system is represented by q second-order differential equations corresponding to three librational and $q-3$ vibrational degrees of freedom, depending upon the number of appendages and admissible functions used. Using the conventional procedure, the equations are transferred into $2N$ first-order equations where all degrees of freedom are solved for simultaneously as an initial value problem. For the procedure to succeed, it is necessary to use the latest available data while updating derivatives.

Organization of the computer program is centered on a series of subroutines. A main program directs the integration process, calling the system-supplied routines as desired, and provides the needed input/output services. The integration package requires a routine (SYSTM) to define the system dynamics in terms of explicit expressions for the first-order derivative of the state vector. The governing equations are employed directly in SYSTM in two distinct stages to deal with the librational and vibrational contributions. At every stage the modular approach is adopted to permit assessment of the contribution of the shifting center of mass, changes in rigid body inertia, deployment rates, flexibility, and nonlinearities.

The three-axis program is so organized as to accommodate an arbitrary number of assumed modes and appendages. Even a two-mode representation and six beam type appendages would result in a system of 54 first-order equations. Modal integration coefficients are determined independently by numerical quadrature. Where possible, these integrals are evaluated analytically as well. Integrations involving small appendage lengths are expected to be particularly time-consuming as observed by Misra and Modi.[14] To cope with the relatively small step size demanded by the high frequency oscillations, a two-stage integration procedure is established, thus allowing for a complete change in such parameters once during the course of the integration. The program is coded in FORTRAN using double precision variables throughout.

Concluding Remarks

The paper presents a rather general formulation aimed at studying librational dynamics, stability, and control of a relatively large class of spacecraft during deployment of flexible appendages. The governing nonlinear, nonautonomous, and coupled equations are being programmed in a modular fashion to help isolate and evaluate effects of the shifting center of mass, appendage offset, changes in central rigid body inertia during deployment, etc., which complicate the equations considerably. This promises to simplify the equations for a large number of satellites currently being designed with a substantial saving in the computational cost. Studies aimed at related situations such as unfolding of accordion type solar panels, reorientation of an orbiting telescope or onboard armament, construction of space platforms, tethered rescue operations, disposal of nuclear waste, etc., would follow a similar approach.

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References

[1]Shrivastava, S.K., Tschann, C., and Modi, V.J., "Librational Dynamics of Earth Orbiting Satellites—A Brief Review," *Proceedings of the XIVth Congress of Theoretical and Applied Mechanics*, ISTAM Publisher, Kharagpur, India, Dec. 1969, pp. 284-306.

[2]Glaser, P.E., "Solar Power Via Satellite," *Astronautics & Aeronautics*, Vol. 11, Aug. 1973, pp. 60-68.

[3]Covington, C. and Piland, R.O., "Space Operations Center—Next Goal for Manned Flight?" *Astronautics & Aeronautics*, Vol. 18, Sept. 1980, pp. 30-37.

[4]Brodsky, R.F. and Morias, B.G., "Space 2020," *Astronautics & Aeronautics*, Vol. 20, May 1982, pp. 54-73.

[5]Burke, B.F., "Radio Telescopes Bigger than the Earth," *Astronautics & Aeronautics*, Vol. 20, Oct. 1982, pp. 44-52.

[6]Joshi, S.M., "Control System Synthesis for a Large Flexible Space Antenna," 33rd Congress of the International Astronautical Federation, Paper No. 82-320, Paris, France, Sept.-Oct. 1982.

[7]Likins, P.W., "Dynamics and Control of Flexible Space Vehicles," NASA TR-32-1329, Jan. 1970.

[8]Likins, P.W. and Bouvier, H.K., "Attitude Control of Nonrigid Spacecraft," *Astronautics & Aeronautics*, Vol. 9, May 1971, pp. 64-71.

[9]Modi, V.J., "Attitude Dynamics of Satellites with Flexible Appendages—A Brief Review," *Journal of Spacecraft and Rockets*, Vol. 11, Nov. 1974, pp. 743-751.

[10]Williams, C.J.H., "Dynamics Modelling and Formulation Techniques for Non-Rigid Spacecraft," *Proceedings of the ESA Symposium on Dynamics and Control of Non-Rigid Spacecraft*, ESA SP 117, Frascati, Italy, May 1976, pp. 53-70.

[11]Roberson, R.E., "Two Decades of Spacecraft Attitude Control," *Journal of Guidance and Control*, Vol. 2, Jan.-Feb. 1979, pp. 3-8.

[12]Lips, K.W., "Dynamics of a Large Class of Satellites with Deploying Flexible Appendages," Ph.D. Dissertation, University of British Columbia, Vancouver, B.C., Canada, Sept. 1980.

[13]Markland, C.A., "A Review of the Attitude Control of Communications Satellites," 32nd Congress of the International Astronautical Federation, Paper No. IAF-81-344, Rome, Italy, 1981.

[14]Modi, V.J. and Misra, A.K., "On the Deployment Dynamics of Tether Connected Two-Body Systems," *Acta Astronautica*, Vol. 6, No. 9, 1979, pp. 1183-1197.

[15]Lang, W. and Honeycutt, G.H., "Simulation of Deployment Dynamics of Spinning Spacecraft," NASA TN-D-4074, 1967.

[16]Cloutier, G.J., "Dynamics of Deployment of Extendible Booms from Spinning Space Vehicles," *Journal of Spacecraft and Rockets*, Vol. 5, May 1968, pp. 547-552.

[17]Bowers, Jr., E.J. and Williams, C.E., "Optimization of RAE Satellite Boom Deployment Timing," *Journal of Spacecraft and Rockets*, Vol. 7, Sept. 1970, pp. 1057-1062.

[18]Hughes, P.C., "Dynamics of a Spin-Stabilized Satellite During Extension of Rigid Booms," *CASI Transactions*, Vol. 5, 1972, pp. 11-19.

[19]Sellapan, R. and Bainum, P.M., "Dynamics of Spin-Stabilized Spacecraft During Deployment of Telescoping Appendages," *Journal of Spacecraft and Rockets*, Vol. 13, Oct. 1976, pp. 605-610.

[20]Cherchas, D.B., "Dynamics of Spin-Stabilized Satellites During Extension of Long Flexible Booms," *Journal of Spacecraft and Rockets*, Vol. 8, July 1971, pp. 802-804.

[21]Cherchas, D.B. and Gossain, D.M., "Dynamics of a Flexible Solar Array During Deployment from a Spinning Spacecraft," *CASI Transactions*, Vol. 7, 1974, pp. 10-18.

[22]Dow, P.C., Scammell, F.H., Murray, F.T., Carlson, N.A., and Buck, I.H., "Dynamic Stability of a Gravity Gradient Stabilized Satellite Having Long Flexible Antennas," *Proceedings of the AIAA/JACC Guidance and Control Conference*, New York, 1966, pp. 285-303.

[23]Jankovic, M.S., "Deployment Dynamics of Flexible Spacecraft," Ph.D. Dissertation, Institute of Aerospace Studies, University of Toronto, Toronto, Canada, 1980.

[24]Ibrahim, A.E. and Misra, A.K., "Attitude Dynamics of a Satellite During Deployment of Large Plate-Type Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 5, Sept.-Oct. 1982, pp. 442-447.

[25]Lips, K.W. and Modi, V.J., "General Dynamics of a Large Class of Flexible Satellite Systems," *Acta Astronautica*, Vol. 7, 1980, pp. 1349-1360.